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1995 J. Phys. A: Math. Gen. 28 6335

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Equivalences between stochastic systems

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Received 19 April 1995

Abstract. Time-dependent correlation functions of (unstable) particles undergoing biased or unbiased diffusion, coagulation and annihilation are studied. This is achieved by similarity transformations between different stochastic models and between stochastic and soluble *non-stochastic* models. For special cases we obtain exact results which are in good agreement with experiments on one-dimensional annihilation–coagulation processes.

The physics of interacting particles out of thermodynamic equilibrium has been a subject of much recent interest. While in larger spatial dimensions, conventional rate equation approaches are sufficient, systems constrained to be effectively one-dimensional display novel and interesting fluctuation effects. For example, for particles A diffusing on a lattice and undergoing a binary reaction process $A + A \rightarrow$ products one expects, for large times t , an algebraic fall-off of the mean particle concentration

$$\bar{c}(t) \sim t^{-\gamma}. \quad (1)$$

In 1D systems, one finds $\gamma = \frac{1}{2}$ as opposed to $\gamma = 1$, obtained from a (mean-field) rate equation. The exponent γ has also been measured experimentally in effectively one-dimensional systems. For annihilation–coagulation reactions $A + A \rightarrow$ products one finds $\gamma = 0.52\text{--}0.59$ [1] and $\gamma = 0.47(3)$ [2], and for the (pure) coagulation reaction $A + A \rightarrow A$, $\gamma \simeq 0.48$ [3]. Although these reaction systems might appear to be quite different, in this work we show that these and more general systems can be treated in a simple and unified way. In particular, a simple explanation for the same value of γ in all annihilation–coagulation problems (that is, with *only* the irreversible reactions $A + A \rightarrow \emptyset$ and $A + A \rightarrow A$ present) is obtained.

For the theoretical description of these reaction–diffusion systems, a useful approach [4] consists of rewriting the master equation which describes the time evolution of the probability distribution function $P(\{\beta\}; t)$ as a Schrödinger equation

$$\partial_t P(\{\beta\}; t) = -HP(\{\beta\}; t) \quad (2)$$

in which the quantum Hamiltonian H is defined in terms of the transition rates $w(\beta \rightarrow \gamma)$ between two configurations $\{\beta\}$ and $\{\gamma\}$ by

$$\langle \gamma | H | \beta \rangle = -w(\beta \rightarrow \gamma) \quad \langle \beta | H | \beta \rangle = \sum_{\gamma \neq \beta} w(\beta \rightarrow \gamma) \quad (3)$$

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where H describes a stochastic process since the sums of all elements in each column add up to zero. This conservation of probability is equivalent to the relation

$$\langle s | H = 0 \quad \langle s | = \sum_{\beta} \langle \{\beta\} | \quad (4)$$

for the vector $\langle s |$. Then the well known theorems about the solutions of the master equation [5] can be translated into the Hamiltonian formulation at hand. In particular, the real parts of the eigenvalues of H are non-negative. Furthermore, starting from an initial probability distribution $|P_0\rangle = \sum_{\beta} P(\{\beta\}; t=0) |\{\beta\}\rangle$ where each configuration $\{\beta\}$ occurs with probability $P(\{\beta\}; t=0)$, the solution to the master equation (2) is then formally given by the time-dependent probability distribution (state vector) $|P\rangle = \sum_{\beta} P(\{\beta\}; t) |\{\beta\}\rangle = \exp(-Ht) |P_0\rangle$. Time-dependent averages of an observable† F are given by the matrix element

$$\langle F \rangle(t) = \langle s | F e^{-Ht} | P_0 \rangle. \quad (5)$$

The interest in this set-up of the problem in 1D comes from the integrability of the quantum Hamiltonian H for large classes of reaction-diffusion processes [6-8]. Exactly known results such as the knowledge of the spectrum of H (obtained, e.g. from the Bethe ansatz) have so far led to a number of exact and explicit results for time-dependent averages and correlations. Here we show how these and other existing results (see, e.g. [9, 10] for annihilation and coagulation processes) can be extended to considerably more general quantum Hamiltonians:

(i) We investigate relations between stochastic systems whose quantum Hamiltonians H and \tilde{H} are related through a change of basis of the one-particle states, see [6-8, 11, 12]

$$\tilde{H} = B H B^{-1} \quad B = \bigotimes_{i=1}^L B_i \quad (6)$$

where B_i is acting only on the site i .

(ii) We consider a given quantum (and, in general, non-stochastic) Hamiltonian with known properties and we look for stochastic processes which can be obtained from this Hamiltonian by a similarity transformation of the form (6). The time-dependent behaviour of these new stochastic systems can then be elucidated in terms of the original Hamiltonian.

Under the similarity transformation B from (6) averages transform as follows:

$$\langle F \rangle(t) = \langle s | F e^{-Ht} | P_0 \rangle = \langle s | \tilde{F} e^{-\tilde{H}t} | \tilde{P}_0 \rangle \quad (7)$$

with the transformed observable $\tilde{F} = F B^{-1}$ and transformed initial distribution $|\tilde{P}_0\rangle = B |P_0\rangle$.

We now give the general form of the single-site matrix B for transformations between two stochastic systems. Certainly, one-site state vectors of the system S described by the Hamiltonian H must have the form $|\rho\rangle = \begin{pmatrix} 1-\rho \\ \rho \end{pmatrix}$, with $0 \leq \rho \leq 1$. Also, for the transformed state $B|\rho\rangle$, probabilities must sum up to one, for all values of ρ , thus

$$b_{11}(1-\rho) + b_{21}\rho + b_{12}(1-\rho) + b_{22}\rho = 1 \quad (8)$$

where the b_{ij} are the elements of B . Comparing coefficients, we get for B the form

$$B = \begin{pmatrix} b_1 & 1-b_2 \\ 1-b_1 & b_2 \end{pmatrix}. \quad (9)$$

Since obviously $\langle s | B = \langle s |$, it follows that $\langle s | \tilde{H} = 0$ for the full system defined on L sites.

† F is a suitably chosen projection operator (see below).

The stochastic systems $S = (H, \rho)$ under consideration are described by the quantum Hamiltonian H and a set of parameters ρ specifying the initial conditions (see below). The relations between two stochastic systems S and \tilde{S} are caught by the following two definitions:

(i) The transformation $S \rightarrow \tilde{S}$ between two stochastic systems S and \tilde{S} is called a *similarity transformation*, if there exists a non-singular transformation B of the form (6) between the quantum Hamiltonians H and \tilde{H} such that all reaction-diffusion rates are positive in both systems. S and \tilde{S} are then called *similar*.

(ii) The transformation $S \rightarrow \tilde{S}$ between two systems S and \tilde{S} is called a *stochastic similarity transformation (SST)*, if S and \tilde{S} are similar and furthermore if for all probability distributions $|P\rangle$ of S also $|\tilde{P}\rangle = B|P\rangle$ is a probability distribution of \tilde{S} .

Through similarities and SST a given system may be mapped into a simpler form. Examples will be given below. To illustrate the second definition, consider an uncorrelated initial state of the form

$$|P_0\rangle = \bigotimes_{i=1}^L \begin{pmatrix} 1 - \rho_i \\ \rho_i \end{pmatrix} \tag{10}$$

with $0 \leq \rho_i \leq 1$ for all sites $i = 1, \dots, L$. This also specifies the initial condition which is going to be considered below, with $\rho_i = \rho$ for all sites i . If \tilde{S} is obtained from S by an SST, initial states of the form (10) are mapped onto transformed initial states $|\tilde{P}_0\rangle$ of the same form and with $0 \leq \tilde{\rho}_i \leq 1$ for all sites. We stress that the notion of an SST between two stochastic systems S and \tilde{S} is considerably more restrictive than mere similarity, which does not also require that the transformed state vector is a probability distribution. We also remark that because of the locality of the change of basis the results obtained here are valid in an arbitrary number of space dimensions, although we shall present the argument explicitly only for $d = 1$.

We now define precisely the models we shall study below. Consider a one-dimensional lattice, with L sites and periodic boundary conditions. Each lattice site can either be empty (denoted by \emptyset) or occupied by a single particle (denoted by A). Particles can hop to an empty nearest-neighbour site. A single particle or a pair of particles on neighbouring sites can undergo a chemical reaction. The reactions we are going to consider are specified with their rates in table I.

Table 1. Two-sites reaction-diffusion processes and their rates.

Diffusion to the left	$\emptyset + A \rightarrow A + \emptyset$	D_L
Diffusion to the right	$A + \emptyset \rightarrow \emptyset + A$	D_R
Pair annihilation	$A + A \rightarrow \emptyset + \emptyset$	2α
Coagulation to the right	$A + A \rightarrow \emptyset + A$	γ_R
Coagulation to the left	$A + A \rightarrow A + \emptyset$	γ_L
Death	$A + \emptyset \rightarrow \emptyset + \emptyset$	δ
	$\emptyset + A \rightarrow \emptyset + \emptyset$	δ
Decoagulation to the left	$\emptyset + A \rightarrow A + A$	β_L
Decoagulation to the right	$A + \emptyset \rightarrow A + A$	β_R
Birth	$\emptyset + \emptyset \rightarrow A + A$	2ν
Creation	$\emptyset + \emptyset \rightarrow \emptyset + A$	σ
	$\emptyset + \emptyset \rightarrow A + \emptyset$	σ

Furthermore, we shall distinguish between unbiased and biased reactions. For unbiased reactions, the rates with indices L and R are all equal and we shall then drop the index

(e.g. $D_L = D_R = D$ etc). For biased reactions, we define an anisotropy parameter η from

$$\begin{aligned} D_L &= D(1 + \eta) & \gamma_L &= \gamma(1 + \eta) & \beta_L &= \beta(1 + \eta) \\ D_R &= D(1 - \eta) & \gamma_R &= \gamma(1 - \eta) & \beta_R &= \beta(1 - \eta). \end{aligned} \quad (11)$$

For $\eta = 0$ we recover the unbiased case.

We begin with unbiased systems, that is $\eta = 0$. We consider the following system, with diffusion, annihilation, coagulation and death reactions present (see table 1). In what follows, we take units of time such that $D = 1$. Then, following the standard procedure of [6] the Hamiltonian can be written in terms of two-site contributions $H = \sum_i H_{i,i+1}$, where

$$H_{i,i+1} = \begin{pmatrix} 0 & -\delta & -\delta & -2\alpha \\ 0 & 1 + \delta & -1 & -\gamma \\ 0 & -1 & 1 + \delta & -\gamma \\ 0 & 0 & 0 & 2(\alpha + \gamma) \end{pmatrix} \quad (12)$$

acts on sites i and $i + 1$ of the lattice. We define the k -point correlation function of the particle number operator n_x , $i = 1, 2, \dots, k$ as

$$C_k(\{x\}, t; \alpha, \gamma, \delta; \rho) = \langle n_{x_1} \dots n_{x_k} \rangle_H(t) \quad (13)$$

where we explicitly indicate the dependence on the rates as well as on the initial conditions. The operator n_x is a projector with eigenvalue 1 if site x is occupied and eigenvalue 0 if it is vacant. Although H is non-Hermitian, it is known that there is a decomposition $H = H_{XXZ} + H_1$ into a Hermitian Hamiltonian H_{XXZ} (which is the Hamiltonian of the anisotropic Heisenberg quantum spin model) and a non-Hermitian part H_1 such that the eigenvalues of H are exactly the eigenvalues of H_{XXZ} [6]. That is so because the chemical reactions permitted here only destroy and never create particles. In one dimension, an interesting special case is given by the *free-fermion* condition

$$\alpha + \gamma = 1 + \delta. \quad (14)$$

In that case the Hermitian part H_{XXZ} can be diagonalized exactly in terms of free fermions. If either just annihilation or coagulation are present, it is known that a closed system of equations of motion can be found [9]. Equation (14) means that diffusion and death together occur at the same rate as annihilation and coagulation together. If $\delta \leq \gamma$, we can rewrite the problem as an annihilation-coagulation problem of an unstable particle, where the effective coagulation rate is modified into $\gamma_{\text{eff}} = \gamma - \delta$, and $1/(2\delta)$ is the lifetime of the unstable particle†. If we use the diffusion process to determine the time scale, we can say that if two particles attempt to be on the same site at the same time, they undergo a chemical reaction with probability one. The ratio $\gamma_{\text{eff}}/\alpha$ then gives the branching ratio between the two processes.

At first sight, the condition (14) appears to be rather artificial. However, it is apparently realized to a good approximation in one of the experimental realizations of the model considered so far [3]. The carrier substance is $\text{N}(\text{CH}_3)_4\text{MnCl}_3$. The particles are excitons of the Mn^{2+} ion and move along the widely separated MnCl_3 chains. A single exciton has a decay time of about 0.7 ms. The on-chain hopping rate is 10^{11} – 10^{12} s⁻¹. If two excitons arrive on the same Mn^{2+} ion, they undergo a coagulation reaction with a reaction time ≈ 100 fs [3]. Since the reaction time is much smaller than the diffusion time, we can conclude that the reaction probability is very close to one. This gives back (14), with $\alpha = \delta = 0$ for this example.

† The special case $\gamma = \delta$, i.e. pair annihilation of unstable particles, is discussed in the appendix.

After these preparatory remarks we return to the general case. Starting from the system S as defined through its Hamiltonian (12), we get the following simplified systems \tilde{S} through an SST.

I. $\delta > 2\alpha + \gamma$. Through an SST we get the system \tilde{S}_I with

$$\tilde{D} = D \quad \tilde{\alpha} = 0 \quad \tilde{\gamma} = \alpha + \gamma \quad \tilde{\delta} = \delta \quad \tilde{\rho} = \frac{\delta - 2\alpha - \gamma}{\delta - \alpha - \gamma} \rho \quad (15)$$

and we find

$$C_k(\{x\}, t; \alpha, \gamma, \delta; \rho) = \left(\frac{\delta - \alpha - \gamma}{\delta - 2\alpha - \gamma} \right)^k C_k(\{x\}, t; 0, \alpha + \gamma, \delta; \frac{\delta - 2\alpha - \gamma}{\delta - \alpha - \gamma} \rho). \quad (16)$$

II. $\delta = 2\alpha + \gamma$. In this case, the transformation becomes singular. However, the equations of motion for the particle number correlators $C(\{x\})$ decouple from each other [8]. For example, the particle density at time t is

$$C_1(x, t) = e^{-2(1+\delta)t} \sum_{m=-\infty}^{\infty} \sum_{y=1}^L C_1(y, 0) I_{x-y+mL}(2t) \quad (17)$$

where I_n is a modified Bessel function.

III. $\delta < 2\alpha + \gamma$. We find the transformed system \tilde{S}_{III}

$$\tilde{D} = D \quad \tilde{\alpha} = \alpha + \gamma \quad \tilde{\gamma} = 0 \quad \tilde{\delta} = \delta \quad \tilde{\rho} = \frac{2\alpha + \gamma - \delta}{2\alpha + 2\gamma - \delta} \rho \quad (18)$$

and we find

$$C_k(\{x\}, t; \alpha, \gamma, \delta; \rho) = \left(\frac{2\alpha + 2\gamma - \delta}{2\alpha + \gamma - \delta} \right)^k C_k(\{x\}, t; \alpha + \gamma, 0, \delta; \frac{2\alpha + \gamma - \delta}{2\alpha + 2\gamma - \delta} \rho). \quad (19)$$

IV. $\delta < \alpha + \gamma$. In this interval, there is a second SST onto the system \tilde{S}_{IV} with

$$\tilde{D} = D \quad \tilde{\alpha} = \alpha + \gamma - \delta \quad \tilde{\gamma} = \delta \quad \tilde{\delta} = \delta \quad \tilde{\rho} = \frac{2\alpha + \gamma - \delta}{2\alpha + 2\gamma - 2\delta} \rho \quad (20)$$

and we find

$$C_k(\{x\}, t; \alpha, \gamma, \delta; \rho) = \left(\frac{2\alpha + 2\gamma - 2\delta}{2\alpha + \gamma - \delta} \right)^k C_k(\{x\}, t; \alpha + \gamma - \delta, \delta, \delta; \frac{2\alpha + \gamma - \delta}{2(\alpha + \gamma - \delta)} \rho). \quad (21)$$

Although both systems \tilde{S}_{III} , \tilde{S}_{IV} are found from S through an SST, the latter is more useful for practical calculations. Using the results derived in the appendix, we can isolate the dependence on δ completely. For translationally invariant initial distributions we have for the large-time behaviour in the free fermion case (see the appendix for more general cases)

$$C_1(x, t) \simeq \tilde{\rho}_0 e^{-2\delta t} \quad (22)$$

$$C_2(x, x+r, t) \simeq t^{-3/2} e^{-4\delta t} \quad r^2 \ll t$$

and we see explicitly that for $\delta \neq 0$ the initial particle density does enter into the large-time behaviour.

This is different from the result found when the death reaction is absent ($\delta = 0$). In that case only the SST onto $\tilde{S}_{III} = \tilde{S}_{IV}$ remains. (The corresponding similarity transformations have been derived before [11, 12].) All correlations then depend non-trivially only on the effective annihilation rate $\alpha_{\text{eff}} = \alpha + \gamma$. The correlation function C on the right-hand

side of (21) is then the known density correlation function for diffusion annihilation, see [6, 9, 10]. For example, with the initial state (10) with $\rho_i = \rho$ for all sites i we have for the mean particle concentration $\bar{c}(t) \sim \int dx C_1(x, t)$ in the free-fermion case for the process $A + A \rightarrow \emptyset$

$$\bar{c}(t) = \rho e^{-4Dt} \left[I_0(4Dt) + 2(1 - \rho) \sum_{k=1}^{\infty} (1 - 2\rho)^{k-1} I_k(4Dt) \right] \simeq (8\pi Dt)^{-1/2} (1 + \mathcal{O}(t^{-1})). \quad (23)$$

In particular, we always get back $y = \frac{1}{2}$ in (1), in agreement with experiment [1–3]. Furthermore, the data of Kroon *et al* [3] show that the long-time behaviour of $\bar{c}(t)$ is *independent* of the initial particle density ρ , in agreement with (23).

So far, the transformations considered have mapped S back onto itself, up to changed values of its parameters. But it is sometimes possible to reduce more complex systems to the ones discussed here. For example, the system S with the parameters

$$D = 1 \quad \delta = 2\alpha + 2\gamma \quad (24)$$

and $\gamma \neq 0$ gives through an SST the system \tilde{S} with (see table 1)

$$\tilde{D} = \frac{1}{3}(3 + 2\alpha + 2\gamma) \quad \tilde{\alpha} = 0 \quad \tilde{\gamma} = \frac{1}{3}(\alpha + \gamma) \quad \tilde{\delta} = 2\tilde{\gamma} \quad \tilde{\nu} = 4\tilde{\gamma}. \quad (25)$$

Since the system \tilde{S} contains particle creation as well as particle destruction terms, it no longer has a trivial (i.e. empty) steady state. This steady state can be found easily, since for a single-site state

$$B \begin{pmatrix} 1 - \rho \\ \rho \end{pmatrix} = \begin{pmatrix} \frac{1}{3} - \frac{\gamma}{3\alpha + 3\gamma} \rho \\ \frac{2}{3} + \frac{\gamma}{3\alpha + 3\gamma} \rho \end{pmatrix}. \quad (26)$$

Since the steady state of S is just $\otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, we find that the steady state of \tilde{S} has a mean particle density $\bar{\rho} = \frac{2}{3}$. The approach towards this steady state is exponential.

Another example is found when $\alpha + \gamma < \delta < 2\alpha + \gamma$. Then S is similar to \tilde{S} with

$$\begin{aligned} \tilde{D} &= \frac{2}{3}(\delta - \alpha - \gamma) + 1 & \tilde{\alpha} &= \frac{4}{3}(\delta - \alpha - \gamma) \\ \tilde{\gamma} &= \tilde{\delta} = \tilde{\sigma} = \frac{2}{3}(2\alpha + 2\gamma - \delta) & \tilde{\beta} &= \frac{1}{3}\delta \end{aligned} \quad (27)$$

and the one-site state changes into

$$B \begin{pmatrix} 1 - \rho \\ \rho \end{pmatrix} = \begin{pmatrix} \frac{2}{3} + \frac{\delta - 2\alpha - \gamma}{3(\delta - \alpha - \gamma)} \rho \\ \frac{1}{3} - \frac{\delta - 2\alpha - \gamma}{3(\delta - \alpha - \gamma)} \rho \end{pmatrix} \quad (28)$$

and we get a steady state particle density of $\bar{\rho} = \frac{1}{3}$. The approach towards this steady state is exponential. The transformation $\tilde{S} \rightarrow S$ is an SST if $\delta > \frac{4}{3}\alpha + \gamma$. Conversely, $S \rightarrow \tilde{S}$ is an SST if $\delta < \frac{4}{3}\alpha + \gamma$. Other examples with $\delta = 0$ are given in [12].

We now turn our attention to some systems with *biased* reaction–diffusion processes. We take diffusion, coagulation and annihilation into account. The rates are given in (11). Using the unbiased case $\eta = 0$ as a guide, we seek an SST $S \rightarrow \tilde{S}$ such that $\tilde{\gamma}_{L,R} = 0$. In fact, using $b_1 = 1$ as before and choosing b_2 in order to get $\tilde{\gamma}_R = 0$, we find

$$\tilde{\gamma}_L = \frac{4(\alpha + \gamma - 1)\gamma\eta_-}{2\alpha + \gamma + \eta(\gamma - 2)} \quad \tilde{\eta} = \eta \quad \tilde{D} = D = 1. \quad (29)$$

For $\eta = 0$, we recover the previous result. However, if we use the free-fermion condition $\alpha + \gamma = 1$, then $\tilde{\gamma}_L = 0$ and $\tilde{\alpha} = 1$. We then have

$$C_k(\{x\}, t; \alpha, \gamma = 1 - \alpha; \eta; \rho) = \left(\frac{2}{1 + \alpha}\right)^k C_k(\{x\}, t; 1, 0; \eta; \frac{1}{2}(1 + \alpha)\rho). \tag{30}$$

Considering the mean particle density only, this relation was also observed in [13] for the special case of mapping the pure biased coagulation problem ($\alpha = 0$) onto the pure biased annihilation problem ($\gamma = 0$). Generally one finds that for a homogeneous initial condition (10) with $\rho_i \equiv \rho = \text{constant}$, the correlation function $C_k(\{x\}, t; 1, 0; \eta; (1 + \alpha)\rho/2)$ is independent of the bias η [10]. For an inhomogeneous initial state with $\rho_{x_0} = 1$ and $\rho_y = \frac{1}{2}$ for $y \neq x_0$ one finds for the average density in an infinite system for large times [10]

$$C_1(x, t) = \langle n_x \rangle(t) = \frac{1}{\sqrt{4\pi t}} + \frac{1}{\pi t^2} e^{-(x-x_0-\eta t)^2/2t}. \tag{31}$$

Finally, we illustrate the transformation (6) between a stochastic and a non-stochastic system. As an example, consider the Hamiltonian $H = \sum_i H_{i,i+1}$

$$H_{i,i+1} = \begin{pmatrix} A & 0 & 0 & -2a \\ 0 & A - 1 & -D & 0 \\ 0 & -D & A - 1 & 0 \\ 0 & 0 & 0 & A - 2 \end{pmatrix}. \tag{32}$$

H can be solved in terms of free fermions. We want the transformed Hamiltonian \tilde{H} to describe a stochastic system, that is we require that $\langle s | \tilde{H} = 0$, see (4). Writing $B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$, the solution to this condition is

$$b_{21} = -b_{11} \quad A = 2. \tag{33}$$

We now take $b_{12} = b_{22}$. Let $\Gamma = (b_{11}/b_{22})^2 > 0$. Then the positivity of the reaction rates in \tilde{H} requires that $\Gamma = D/a$. The Hamiltonian then reads

$$\tilde{H}_{i,i+1} = \begin{pmatrix} 2 - 2D & -1 - D & -1 - D & 0 \\ -1 + D & 2 + 2D & 0 & D - 1 \\ -1 + D & 0 & 2 + 2D & D - 1 \\ 0 & -1 - D & -1 - D & 2 - 2D \end{pmatrix}. \tag{34}$$

The off-diagonal elements of \tilde{H} are non-positive provided $0 < D \leq 1$ and $0 < a$. Under these conditions B is non-singular and \tilde{H} is the quantum Hamiltonian of a stochastic system. We point out that this Hamiltonian is identical to the quantum Hamiltonian for the 1D Glauber–Ising model [14]

$$H_{GI} = \sum_{i=1}^L (1 - \sigma_i^x) \left(1 - \frac{1}{2} D (\sigma_i^z \sigma_{i+1}^z + \sigma_{i-1}^z \sigma_i^z)\right) \tag{35}$$

at temperature T given by $D = \tanh(2J/k_B T)$. In this way we obtain a new relation between non-zero temperature Glauber dynamics and the XXZ chain in the free fermion case. On the other hand, for non-vanishing T the Glauber–Ising model can be transformed into an XY free fermion chain [15]. We shall return to consequences of this observation and the reformulation of more general stochastic processes in terms of soluble free fermion systems elsewhere.

In summary, we have shown how to relate different stochastic systems using similarity transformations. In several examples, this technique proves to be useful to extend the scope of integrable systems. The results are in agreement with the existing experiments

and include some previous observations of relations between different systems as special cases. Going beyond similarity transformations between stochastic systems, we have found a simple example how to reformulate a stochastic system in terms of a non-stochastic soluble free-fermion model in a novel way. The techniques developed here can be used for a systematic study and classification of stochastic interacting particle systems in terms of integrable quantum chains.

Acknowledgments

We thank K Krebs and H Simon for useful correspondence. MH, EO and GMS were supported by the EC 'Human Capital and Mobility' programme.

Appendix

Here we study the annihilation process for unstable particles with an average lifetime $\gamma = \delta$. The equations of motion for the average density and density correlations read

$$\frac{d}{dt}\langle n_x \rangle = \langle n_{x+1} \rangle + \langle n_{x-1} \rangle - 2(1 + \delta)\langle n_x \rangle - 2\alpha (\langle n_x n_{x+1} \rangle + \langle n_{x-1} n_x \rangle) \quad (\text{A1})$$

$$\begin{aligned} \frac{d}{dt}\langle n_x n_y \rangle &= \langle n_{x+1} n_y \rangle + \langle n_{x-1} n_y \rangle + \langle n_x n_{y+1} \rangle + \langle n_x n_{y-1} \rangle - 4(1 + \delta)\langle n_x n_y \rangle \\ &\quad - 2\alpha (\langle n_x n_{x+1} n_y \rangle + \langle n_{x-1} n_x n_y \rangle + \langle n_x n_y n_{y+1} \rangle + \langle n_x n_y n_{y-1} \rangle) \\ &\quad (\text{if } |y - x| \geq 2) \end{aligned} \quad (\text{A2})$$

$$\begin{aligned} \frac{d}{dt}\langle n_x n_{x+1} \rangle &= \langle n_{x-1} n_{x+1} \rangle + \langle n_x n_{x+2} \rangle - 2(1 + \alpha + 2\delta)\langle n_x n_{x+1} \rangle \\ &\quad - 2\alpha (\langle n_{x-1} n_x n_{x+1} \rangle + \langle n_x n_{x+1} n_{x+2} \rangle) \end{aligned} \quad (\text{A3})$$

and similar expressions for higher-order correlators [8]. For a k -point correlator there is always a coupling to $(k + 1)$ -point correlation functions proportional to the annihilation rate α . First we show that the amplitude of a k -point density correlation function $C_k(\{x\}; t) = \langle n_{x_1} \dots n_{x_k} \rangle$ decays for large times t at least with a factor proportional to $\exp(-2k\delta t)$.

To see this, we recall that the spectrum of the quantum Hamiltonian H is exactly the same as of the XXZ quantum chain [6]. For the calculation of the eigenvalues, it is thus sufficient to consider the sectors with fixed number of particles k separately. In the k -particle sector, the eigenvalues of H are

$$E_{\{|i\}} = k2\delta + \sum_{i=1}^N 2(1 - \cos q_i) \geq k2\delta \quad (\text{A4})$$

and the values of the q_i are determined from the Bethe ansatz equations. On the other hand, the k -point correlators C_k can only take non-zero values when defined on states which contain at least k particles. Furthermore, C_k depends through the equations of motion directly only on C_k and C_{k+1} and in particular it is independent of C_0, C_1, \dots, C_{k-1} . Thus, for large times t we must have

$$C_k(\{x\}; t) \sim e^{-\lambda t} \quad \text{with } \lambda \geq k2\delta \quad (\text{A5})$$

because of the inequality (A4) for the eigenvalues of H in the k -particle sector[†].

[†] Writing $C_k(\{x\}; t) = e^{-2k\delta t} B_k(\{x\}; t)$, one can further show with the Bethe ansatz that generically $B_1(t) \rightarrow \mathcal{O}(1)$ and $B_{k \geq 2}(t) \rightarrow 0$ as $t \rightarrow \infty$.

Now we define the explicitly time-dependent quantity $\tilde{n}_x(t) = \exp(2\delta t)n_x$. According to the considerations of the previous paragraph k -point correlation functions of this quantity are bounded from above by their initial values $\langle \tilde{n}_{x_1} \dots \tilde{n}_{x_k} \rangle(t=0) = \langle n_{x_1} \dots n_{x_k} \rangle(t=0) \leq 1$. Rewriting equations (A1)–(A3) and the corresponding equations for k -point correlators in terms of averages for \tilde{n}_{x_i} reproduces equations of the same form, but with effective time-dependent couplings $\tilde{\alpha}(t) = \alpha \exp(-2\delta t)$ to $(k+1)$ -point correlators and with $\tilde{\delta} = 0$:

$$\frac{d}{dt} \langle \tilde{n}_x \rangle = \langle \tilde{n}_{x+1} \rangle + \langle \tilde{n}_{x-1} \rangle - 2 \langle \tilde{n}_x \rangle - 2\alpha e^{-2\delta t} (\langle \tilde{n}_x \tilde{n}_{x+1} \rangle + \langle \tilde{n}_{x-1} \tilde{n}_x \rangle) \quad (\text{A6})$$

$$\begin{aligned} \frac{d}{dt} \langle \tilde{n}_x \tilde{n}_y \rangle &= \langle \tilde{n}_{x+1} \tilde{n}_y \rangle + \langle \tilde{n}_{x-1} \tilde{n}_y \rangle + \langle \tilde{n}_x \tilde{n}_{y+1} \rangle + \langle \tilde{n}_x \tilde{n}_{y-1} \rangle - 4 \langle \tilde{n}_x \tilde{n}_y \rangle \\ &\quad - 2\alpha e^{-2\delta t} (\langle \tilde{n}_x \tilde{n}_{x+1} \tilde{n}_y \rangle + \langle \tilde{n}_{x-1} \tilde{n}_x \tilde{n}_y \rangle + \langle \tilde{n}_x \tilde{n}_y \tilde{n}_{y+1} \rangle + \langle \tilde{n}_x \tilde{n}_{y-1} \tilde{n}_y \rangle) \\ &\quad (\text{if } |y-x| \geq 2) \end{aligned} \quad (\text{A7})$$

$$\begin{aligned} \frac{d}{dt} \langle \tilde{n}_x \tilde{n}_{x+1} \rangle &= \langle \tilde{n}_{x-1} \tilde{n}_{x+1} \rangle + \langle \tilde{n}_x \tilde{n}_{x+2} \rangle - 2(1+\alpha) \langle \tilde{n}_x \tilde{n}_{x+1} \rangle \\ &\quad - 2\alpha e^{-2\delta t} (\langle \tilde{n}_{x-1} \tilde{n}_x \tilde{n}_{x+1} \rangle + \langle \tilde{n}_x \tilde{n}_{x+1} \tilde{n}_{x+2} \rangle). \end{aligned} \quad (\text{A8})$$

Since $\langle \tilde{n}_{x_1} \dots \tilde{n}_{x_k} \rangle(t) \leq 1$ for all times t and since for long times $\alpha e^{-2\delta t}$ can be neglected these equations effectively decouple and reduce to closed linear differential-difference equations for k -point correlators which can be solved with the Bethe ansatz [8].

The equation for the one-point function reduces to a lattice diffusion equation which is solved by modified Bessel functions. Assuming a translationally invariant initial state with $\langle n_x(t=0) \rangle = \langle \tilde{n}_x(t=0) \rangle = \rho_0$ one finds for the average particle density at time t

$$\langle n_x(t) \rangle \approx \rho_0 e^{-2\delta t}. \quad (\text{A9})$$

This quantity depends on the initial density which is in contrast to the diffusion limited annihilation of stable particles where the density decays algebraically (for long times) and with an amplitude independent of the initial density.

Defining $\tilde{C}(r, t) = \langle \tilde{n}_x \tilde{n}_{x+r}(t) \rangle$ and using (A6)–(A8) one finds for the two-point function with translationally invariant initial conditions the equations

$$\begin{aligned} \frac{d}{dt} \tilde{C}(r, t) &= 2(\tilde{C}(r+1, t) + \tilde{C}(r-1, t) - 2\tilde{C}(r, t)) \quad (r \geq 2) \\ \frac{d}{dt} \tilde{C}(1, t) &= 2(\tilde{C}(2, t) - (1+\alpha)\tilde{C}(1, t)). \end{aligned} \quad (\text{A10})$$

This gives for the two-point density correlation function $C(r, t) = \langle n_x n_{x+r}(t) \rangle$ in an infinite system

$$C(r, t) \approx e^{-4(1+\delta)t} \sum_{y=1}^{\infty} [a_y I_{r-y}(4t) + b_y I_{r+y-1}(4t)] \quad (\text{A11})$$

where I_n is the modified Bessel function, a_y are constants defined by the initial distribution and $b_y = \mu a_y - (1-\mu^2) \sum_{k=1}^{y-1} \mu^{y-1-k} a_k$ with $\mu = 1-\alpha$. At first sight, we should expect for large times that $C(r, t) \sim t^{-1/2} \exp(-4\delta t)$. For the free-fermion case $\alpha = 1$, however, a different result is found. Since $b_y = -a_{y-1}$ we get

$$C(r, t) \approx e^{-4(1+\delta)t} \sum_{y=1}^{\infty} a_y [I_{r-y}(4t) - I_{r+y}(4t)]. \quad (\text{A12})$$

For $r^2 \ll t$ this correlator decays as $C \sim t^{-3/2} \exp(-4\delta t)$ whereas for $r^2 \sim t$ one has $C \sim t^{-1/2} \exp(-4\delta t)$. The same effect is also seen for $\alpha = 2$.

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